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Form and index of Ginsparg-Wilson fermions

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Abstract: We clarify the questions rised by a recent example of a lattice Dirac operator found by Chiu. We show that this operator belongs to a class based on the Cayley transformation and that this class on the finite lattice generally does not admit a nonvanishing index, while in the continuum limit, due to operator properties in Hilbert space, this defect is no longer there. Analogous observations are made for the chiral anomaly. We also elaborate on various aspects of the underlying sum rule for the index.

Keywords: Lattice Quantum Field Theory, Lattice Gauge Field Theories.

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1. Introduction

Recently Chiu [1] reported about a lattice Dirac operator, obeying the Ginsparg-Wilson (GW) relation [2] and a mild condition on gauge field configurations, which does not admit a nonvanishing index, while everything else, in particular also the chiral anomaly, appears correct. The derivation of this was based on the sum rule [3] for the index. This remarkable observation rises generally the questions of what might be missing in the concept and how the apparent paradox resolves. To clarify these issues is the aim of the present paper.

In section 2, inspecting possible forms of unitary operators V which define GW Dirac operators, we note that the one implicit in ref. [1] is a Cayley transform. In section 3 we consider the general sum rule for the index and derive various relations which we need in the following. In section 4 we establish the validity of the sum rule also in Hilbert space. In section 5 we point out the differences to the Atiyah-Singer framework. In section 6 we generally analyze Cayley-type forms of V on the finite lattice as well as in the limit and completely clarify the issues in question. Section 7 contains conclusions and summary.

2. Forms of GW fermions

For lattice Dirac operators D which satisfy the GW relation

$$\{\gamma_5, D\} = \rho^{-1} D\gamma_5 D \tag{2.1}$$

and are γ_5-hermitian

$$D^\dagger = \gamma_5 D\gamma_5 \tag{2.2}$$

due to (2.2) the remaining content of (2.1) is $\rho(D+D^\dagger) = D^\dagger D = DD^\dagger$. The latter relation is the condition for unitarity of the operator $V = \mathbb{1} - \rho^{-1}D$, which is also γ_5-hermitian because D is. Therefore instead of using (2.1) and (2.2) we equivalently require D to have the general form

$$D = \rho(\mathbb{1} - V) \qquad \text{with} \qquad V^\dagger = V^{-1} = \gamma_5 V \gamma_5 \tag{2.3}$$

and a real constant ρ. Since V is unitary, the spectrum of D is on a circle through zero around ρ. Real eigenvalues then are possible at 0 and at 2ρ. Clearly D is normal.

There are three standard constructions of unitary operators V. They are up to constant phase factors given

1. by $X(\sqrt{X^\dagger X}\,)^{-1}$, normalizing an operator X, not requiring particular mathematical properties of X (apart from $X^\dagger X \neq 0$),
2. by the Cayley transform $(Y - i\mathbb{1})(Y + i\mathbb{1})^{-1}$ of a hermitian operator Y,
3. by $\exp(iZ)$ with a hermitian generator Z.

In addition to unitarity one gets γ_5-hermiticity of the constructed operators V by requiring X, iY, iZ to be γ_5-hermitian.

Construction i) is used by Neuberger in the overlap Dirac operator [4] choosing

$$V = -X(\sqrt{X^\dagger X}\,)^{-1} \qquad \text{with} \qquad X = m\mathbb{1} + D_W\,, \quad -2 < m < 0\,, \tag{2.4}$$

where D_W is the massless Wilson-Dirac operator and where only X with $X^\dagger X \neq 0$ are admitted. This form is well known to give the correct results and does not need to be discussed further here.

With respect to Construction ii) we observe that it is actually the basis of what has been introduced by Chiu [1, 5]. In fact, the form $D = a^{-1} D_c(\mathbb{1} + rD_c)^{-1}$ in ref. [1], with an appropriate antihermitian operator D_c and a suitable positive constant r, amounts to putting

$$V = -(Y - i\mathbb{1})(Y + i\mathbb{1})^{-1} \qquad \text{with} \qquad Y = irD_c\,, \tag{2.5}$$

where $\rho^{-1} = 2ar$ relates the notations here and there. After a number of necessary preparations, we shall analyze Construction ii) in Section 6 in a general way and clarify the related questions.

Construction iii) so far has not been applied in the present context. For this one could use an operator Z of the form of Y mentioned above and tune r to cover the spectrum appropriately. There remain, however, problems with the periodicity to be settled. We will not address this case here.

3. Index and spectrum

To obtain the relations needed on the finite lattice, where D operates in a unitary space of finite dimension, we have the spectral representation $D = \sum_k \lambda_k P_k$ of normal D with eigenvalues λ_k and orthogonal projections $P_k = P_k^\dagger$. Using the γ_5-hermiticity of D, it gets the more detailed form

$$D = \sum_{j\,(\mathrm{Im}\,\lambda_j = 0)} \lambda_j (P_j^{(+)} + P_j^{(-)}) + \sum_{j\,(\mathrm{Im}\,\lambda_j > 0)} (\lambda_j P_j^{(\mathrm{I})} + \lambda_j^* P_j^{(\mathrm{II})})\,, \tag{3.1}$$

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and one has $P_1^{(+)} + P_1^{(-)}$ projecting on the eigenspace with $\lambda_1 = 0$. Since the projections satisfy $\gamma_5 P_j^{(\pm)} = P_j^{(\pm)}\gamma_5 = \pm P_j^{(\pm)}$ and $\gamma_5 P_j^{(\mathrm{I})} = P_j^{(\mathrm{II})}\gamma_5$, one gets

$$\mathrm{Tr}(\gamma_5 P_j^{(\pm)}) = \pm N_\pm(\lambda_j)\,, \qquad \mathrm{Tr}(\gamma_5 P_j^{(\mathrm{I})}) = \mathrm{Tr}(\gamma_5 P_j^{(\mathrm{II})}) = 0\,, \tag{3.2}$$

where the trace Tr is in full space and $N_+(\lambda_j)$ and $N_-(\lambda_j)$ denote the dimensions of the right-handed and of the left-handed eigenspace for real eigenvalue λ_j, respectively.

Using (3.1) with (3.2) the index $N_+(0) - N_-(0)$ is obtained from the resolvent $(D - \zeta\,\mathbb{1})^{-1}$ by

$$\lim_{\zeta\to 0} \mathrm{Tr}\left(\gamma_5(-\zeta)(D-\zeta\mathbb{1})^{-1}\right) = N_+(0) - N_-(0) \tag{3.3}$$

and one also finds

$$\lim_{\zeta\to 0} \mathrm{Tr}\left(\gamma_5 D(D-\zeta\,\mathbb{1})^{-1}\right) = \sum_{\lambda_j\neq 0\ \text{real}} \Big(N_+(\lambda_j) - N_-(\lambda_j)\Big)\,. \tag{3.4}$$

Adding up eqs. (3.3) and (3.4) the sum of their l.h.s. gets $\mathrm{Tr}(\gamma_5\mathbb{1}) = 0$ and one obtains

$$0 = N_+(0) - N_-(0) + \sum_{\lambda_j\neq 0\ \text{real}} \Big(N_+(\lambda_j) - N_-(\lambda_j)\Big)\,. \tag{3.5}$$

This shows that to make a nonvanishing index at all possible, one has to require that the spectrum of D in addition to zero allows for at least one further real eigenvalue.

The sum rule found by Chiu in ref. [3] and used in ref. [1] is the special case of (3.5),

$$N_+(0) - N_-(0) + N_+(2\rho) - N_-(2\rho) = 0\,, \tag{3.6}$$

which holds for Dirac operators satisfying the GW relation (2.1). In that case (3.4) simplifies to

$$(2\rho)^{-1}\,\mathrm{Tr}(\gamma_5 D) = N_+(2\rho) - N_-(2\rho)\,. \tag{3.7}$$

In the following it will be more appropriate to have the relations in terms of V, in which case ρ drops out completely. For this we note that by (2.3) the eigenvalues 0 and 2ρ of D correspond to the eigenvalues 1 and -1 of V, respectively. Thus, denoting the dimensions of the right-handed and the left-handed eigenspace at eigenvalue ±1 of V by $\tilde N_+(\pm1)$ and $\tilde N_-(\pm1)$, respectively, the sum rule (3.6) turns into

$$\tilde N_+(1) - \tilde N_-(1) + \tilde N_+(-1) - \tilde N_-(-1) = 0\,, \tag{3.8}$$

requiring that in addition to $+1$ the eigenvalue -1 of V can occur. Further, instead of (3.7) with (3.6), we then have for the index

$$\tilde N_+(1) - \tilde N_-(1) = -\tilde N_+(-1) + \tilde N_-(-1) = \frac{1}{2}\,\mathrm{Tr}(\gamma_5 V)\,. \tag{3.9}$$

In this context it is instructive to note that the spectral representation of $\gamma_5 V$ is

$$\gamma_5 V = P_1^{(+)} - P_1^{(-)} - P_2^{(+)} + P_2^{(-)} + \sum_k (\check P_k^{[+]} - \check P_k^{[-]})\,, \tag{3.10}$$

where $P_1^{(\pm)}$ and $P_2^{(\pm)}$ project onto the eigenspaces of V with eigenvalue $+1$ and -1, respectively, and where $\check P_k^{[\pm]} = \frac{1}{2}(P_k^{(\mathrm{I})} + P_k^{(\mathrm{II})} \pm \mathrm{e}^{i\varphi_k}\gamma_5 P_k^{(\mathrm{I})} \pm \mathrm{e}^{-i\varphi_k}\gamma_5 P_k^{(\mathrm{II})})$ with $0 < \varphi_k < \pi$ (and $[\pm]$ being related to signs of eigenvalues of $\gamma_5 V$).

4. Sum rule in Hilbert space

For the clarification of the questions raised in ref. [1] we have to make sure that the sum rule still holds in the continuum limit. Then the operators act in Hilbert space (as will be discussed in detail in section 6), so that we firstly have to give a definition of the traces Tr in infinite space, too, and secondly to show that a continuous part of the spectrum does not contribute to the sum rule.

The immediate prescription for the Tr-expressions is to consider them as the limit of the respective finite-lattice results. Technically more convenient is summing or integrating up a spectral representation *after* taking Tr of its contributions. Clearly any regularization can be used by which one arrives at the limit of the finite-lattice results.

The Tr-expressions here typically involve differences of operators projecting on right-handed and left-handed space, respectively. This suggests to use the notion of cardinalities, which extends the one of dimensions to the infinite case. It allows to compare infinite spaces directly, using that two sets have the same cardinality if there is a bijective mapping between them. For our purpose we restrict such mappings to ones between subspaces which have different chirality and call the resulting concept specialized cardinality.

A simple example is $\mathrm{Tr}(\gamma_5\,\mathbb{1})=0$, which we have used to derive (3.5). In the infinite case one can decompose the identity operator as $\mathbb{1}=\sum_j\hat{P}_j$, with any orthogonal projections $\hat{P}_j$ having finite trace Tr and being trivial in Dirac space, and define $\mathrm{Tr}(\gamma_5\,\mathbb{1})=:\sum_j\mathrm{Tr}(\gamma_5\hat{P}_j)=0$. The special choice $\hat{P}_n=\mathbb{1}_0\otimes p(n)$, where $p(n)$ is the projection related to lattice site n and $\mathbb{1}_0$ the identity operator in Dirac and gauge field space, allows to make contact to the finite-lattice results. With respect to the general meaning, for $\gamma_5\mathbb{1}=P_{\mathrm{R}}-P_{\mathrm{L}}$ with the projections $P_{\mathrm{R}}=(1/2)(1+\gamma_5)\mathbb{1}$ and $P_{\mathrm{L}}=(1/2)(1-\gamma_5)\mathbb{1}$ a bijective mapping is provided by $T=\gamma_4\mathbb{1}$, which gives $T^\dagger P_{\mathrm{R}}T=P_{\mathrm{L}}$ and $T^\dagger P_{\mathrm{L}}T=P_{\mathrm{R}}$. This indicates that P_{R} and P_{L} project on subspaces which have the same specialized cardinality.

A further example is $(1/2)\,\mathrm{Tr}(\gamma_5V)$ in (3.9). The discrete part of the spectral representation of the operator γ_5V is given by (3.10), while the continuous part does not contribute (as will be shown below). One notes that the subspaces on which $\sum_k\check{P}_k^{[+]}$ and $\sum_k\check{P}_k^{[-]}$ project have the same specialized cardinality. This follows because the projections $\check{P}_k^{[+]}$ and $\check{P}_k^{[-]}$ are one-to-one associated and the subspaces on which such pairs project have both dimension $(1/2)\,\mathrm{Tr}(P_k^{(\mathrm{I})}+P_k^{(\mathrm{II})})$. Thus only the part with $P_1^{(\pm)}$ and $P_2^{(\pm)}$ contributes.

We next address the fact that the spectrum of D now can also have a continuous part. The discrete part of the spectral representation is as before. The continuous part in terms of V is represented by a Stieltjes integral,

$$V_{\mathrm{con}}=\int_{-\pi}^{\pi}\mathrm{e}^{i\varphi}\mathrm{d}E_{\mathrm{con}\,\varphi}=\int_0^{\pi}\mathrm{e}^{i\varphi}\mathrm{d}E_{\mathrm{con}\,\varphi}-\int_0^{\pi}\mathrm{e}^{-i\varphi}\mathrm{d}E_{\mathrm{con}\,-\varphi}\,, \tag{4.1}$$

where the subdivision into two integrals is possible because the projector function $E_{\mathrm{con}\,\varphi}$ is purely continuous. With $V^\dagger=\gamma_5V\gamma_5$ and $E^\dagger_{\mathrm{con}\,\varphi}=E_{\mathrm{con}\,\varphi}$ one gets $\gamma_5E_{\mathrm{con}\,\varphi}=-E_{\mathrm{con}\,-\varphi}\gamma_5$. From this, using that $E_{\mathrm{con}\,\varphi'}E_{\mathrm{con}\,\varphi}=E_{\mathrm{con}\,\varphi}E_{\mathrm{con}\,\varphi'}=E_{\mathrm{con}\,\varphi}$ holds for $\varphi\le\varphi'$, it follows that

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$\mathrm{Tr}(\gamma_5 E_{\mathrm{con}\,\varphi}) = 0$. Therefore we obtain for any function $F(V_{\mathrm{con}})$

$$\mathrm{Tr}\Big(\gamma_5 F(V_{\mathrm{con}})\Big) = \int_{-\pi}^{\pi} F(\mathrm{e}^{i\varphi})\,\mathrm{d}\Big(\mathrm{Tr}(\gamma_5 E_{\mathrm{con}\,\varphi})\Big) = 0 \tag{4.2}$$

and an analogous relation for the continuous part of D. Because the sum rule is based on expressions of type $\mathrm{Tr}\Big(\gamma_5 F(V)\Big)$ or $\mathrm{Tr}\Big(\gamma_5 F(D)\Big)$, it is indeed *not* affected by the occurrence of a continuous spectrum.

5. Differences to the Atiyah-Singer case

The sum rule, which in ref. [1] and in the present paper is seen to be of crucial importance, exhibits the mechanism allowing for a nonvanishing index on the lattice. This mechanism is related to the chiral noninvariance, which is conceptually necessary in quantum field theory, and in its lattice definition is present from the start. Clearly in lattice theory the space structure itself neither does depend on the Dirac operator nor does it get chirally asymmetric.

This is in contrast to what one has in case of the Atiyah-Singer Dirac operator [6], where to admit a nonvanishing index the space structure itself must be allowed to depend on the particular gauge field configuration. Because this appears to be not sufficiently realized, we briefly point out some details.

The definition in the Atiyah-Singer case is based on (Weyl) operators $D_{\mathrm{AS}}^{(+)}$ and $D_{\mathrm{AS}}^{(-)}$ which map from the total right-handed space $\mathcal{E}^{(+)}$ to the total left-handed space $\mathcal{E}^{(-)}$ and back, respectively. It is given in the combined space $\mathcal{E}^{(+)} \oplus \mathcal{E}^{(-)}$ by $D_{\mathrm{AS}} = \hat{D}_{\mathrm{AS}}^{(+)} + \hat{D}_{\mathrm{AS}}^{(-)}$ where $\hat{D}_{\mathrm{AS}}^{(\pm)} = D_{\mathrm{AS}}^{(\pm)}$ on $\mathcal{E}^{(\pm)}$ and $\hat{D}_{\mathrm{AS}}^{(\pm)} = 0$ on $\mathcal{E}^{(\mp)}$. Because of $D_{\mathrm{AS}}^{(+)\dagger} = D_{\mathrm{AS}}^{(-)}$ the Dirac operator D_{AS} is self-adjoint and since it acts on a compact manifold its spectrum is discrete. Thus it is represented by $D_{\mathrm{AS}} = \sum_j \lambda_{\mathrm{AS}\,j} P_{\mathrm{AS}\,j}$ which implies $D_{\mathrm{AS}}^2 = \sum_j \lambda_{\mathrm{AS}\,j}^2 P_{\mathrm{AS}\,j}$.

On the other hand, one gets $D_{\mathrm{AS}}^2 = \hat{D}_{\mathrm{AS}}^{(+)}\hat{D}_{\mathrm{AS}}^{(-)} + \hat{D}_{\mathrm{AS}}^{(-)}\hat{D}_{\mathrm{AS}}^{(+)}$, which can be evaluated noting that the operators $D_{\mathrm{AS}}^{(-)}D_{\mathrm{AS}}^{(+)}$ and $D_{\mathrm{AS}}^{(+)}D_{\mathrm{AS}}^{(-)}$ map within $\mathcal{E}^{(+)}$ and $\mathcal{E}^{(-)}$, respectively, and are selfadjoint and nonnegative. With the eigenequation $D^{(-)}D^{(+)}\Phi_j = \kappa_j\Phi_j$ in $\mathcal{E}^{(+)}$ one gets the eigenequation $D^{(+)}D^{(-)}(D^{(+)}\Phi_j) = \kappa_j(D^{(+)}\Phi_j)$ in $\mathcal{E}^{(-)}$. Further, from $\langle D^{(+)}\Phi_{jr'}|D^{(+)}\Phi_{jr}\rangle = \langle\Phi_{jr'}|D^{(-)}D^{(+)}\Phi_{jr}\rangle = \kappa_j\langle\Phi_{jr'}|\Phi_{jr}\rangle$ one sees that, except for $\kappa_j = 0$, for each of the common eigenvalues κ_j the eigenspaces must have the same dimension. Therefore, except for $\kappa_j = 0$, the operators $D_{\mathrm{AS}}^{(-)}D_{\mathrm{AS}}^{(+)}$ and $D_{\mathrm{AS}}^{(+)}D_{\mathrm{AS}}^{(-)}$ have the same spectra. Denoting the projections on their eigenspaces by $P_{\mathrm{AS}\,j}^{(+)}$ and $P_{\mathrm{AS}\,j}^{(-)}$, respectively, and comparing the above expressions for D_{AS}^2 it then follows that $D_{\mathrm{AS}} = \sum_j \lambda_{\mathrm{AS}\,j}(P_{\mathrm{AS}\,j}^{(+)} + P_{\mathrm{AS}\,j}^{(-)})$ and that one always has

$$\hat{N}_+(\lambda_{\mathrm{AS}\,j}) = \hat{N}_-(\lambda_{\mathrm{AS}\,j}) \qquad \text{for} \qquad \lambda_{\mathrm{AS}\,j} \neq 0 \tag{5.1}$$

for the dimensions $\hat{N}_+(\lambda_{\mathrm{AS}\,j}) = \mathrm{Tr}\,P_{\mathrm{AS}\,j}^{(+)}$ and $\hat{N}_-(\lambda_{\mathrm{AS}\,j}) = \mathrm{Tr}\,P_{\mathrm{AS}\,j}^{(-)}$ of the eigenspaces.

For $\kappa_j = \lambda_{\mathrm{AS}\,j} = 0$ one gets $\ker D_{\mathrm{AS}}^{(+)} = \ker D_{\mathrm{AS}}^{(-)}D_{\mathrm{AS}}^{(+)}$ and $\ker D_{\mathrm{AS}}^{(-)} = \ker D_{\mathrm{AS}}^{(+)}D_{\mathrm{AS}}^{(-)}$ (as is obvious from left to right and follows from right to left from $\langle\Phi|D_{\mathrm{AS}}^{(-)}D_{\mathrm{AS}}^{(+)}\Phi\rangle =$

$\langle D_{\rm AS}^{(+)}\Phi|D_{\rm AS}^{(+)}\Phi\rangle)$. Thus the eigenspaces with eigenvalue zero have the dimensions $\hat{N}_{\pm}(0) = \dim\ker D_{\rm AS}^{(\pm)}$ and the index becomes $\hat{N}_{+}(0) - \hat{N}_{-}(0) = \dim\ker D_{\rm AS}^{(+)} - \dim\ker D_{\rm AS}^{(+)\dagger}$.

It is now seen that because of (5.1), to admit a nonvanishing index $\hat{N}_{+}(0) - \hat{N}_{-}(0)$, the space structure itself must be allowed to vary and to get chirally asymmetric, and thus to depend on the particular gauge field configuration. It also becomes obvious that in case of a nonvanishing index the specialized cardinalities of $\mathcal{E}^{(+)}$ and $\mathcal{E}^{(-)}$ get different. Indeed, while right-handed and left-handed eigenspaces with the same dimension are associated at each $\lambda_{{\rm AS}\,j} \neq 0$, this then does not hold for those with $\lambda_{{\rm AS}\,j} = 0$.

In addition to the fundamentally different space structure, a further difference is that in the Atiyah-Singer framework one considers a differential operator on a compact manifold, while in the definition of quantized theory one deals with a lattice operator and its subtle limit on $\mathbb{R}^4$. Thus altogether special care appears appropriate when considering analogies.

6. Analysis of Cayley-type V

To investigate Construction ii) of section 2 we consider the general choice

$$V = -(Y - i\mathbb{1})(Y + i\mathbb{1})^{-1} = 2(\mathbb{1} + Y^2)^{-1} - \mathbb{1} + i2Y(\mathbb{1} + Y^2)^{-1}\,. \tag{6.1}$$

In (6.1) for the eigenvalue $y = 0$ of Y one obviously gets the eigenvalue $v = 1$ of V.

On the finite lattice the operators act in a unitary space of finite dimension. Therefore, requiring Y to be a well-defined hermitian operator, its spectrum consists of a finite number of real eigenvalues (which are discrete and finite). Introducing $s = \max(|y_{\rm min}|, |y_{\rm max}|)$, where $y_{\rm min}$ and $y_{\rm max}$ denote the smallest and the largest eigenvalue of Y, respectively, (6.1) gives

$$\mathrm{Re}\,v \geq \frac{2}{1+s^2} - 1 \quad \text{for all } v\,, \qquad |\,\mathrm{Im}\,v| \geq \frac{2s}{1+s^2} \quad \text{for } \mathrm{Re}\,v < 0\,. \tag{6.2}$$

We see from this that the eigenvalue $v = -1$ of V cannot be reached. Thus on the finite lattice Construction ii) does generally not meet the basic requirement needed according to (3.8) to allow for a nonvanishing index.

In addition, we observe that , because of not reaching $v = -1$, one always has $P_2^{(\pm)} \equiv 0$ in (3.10). Thus on the finite lattice already the chiral anomaly turns out to be affected.

The obvious obstacle which prevents from reaching the eigenvalue -1 of V is that on the finite lattice Y is bounded. A related problem is that there the inverse Cayley transform,

$$Y = -i(V - \mathbb{1})(V + \mathbb{1})^{-1}\,, \tag{6.3}$$

is not valid for all unitary operators but only for the subset for which the spectrum does not extend to -1.

The crucial observation now is that the indicated restrictions no longer hold in Hilbert space, where one gets a well-defined connection between general unitary operators V and selfadjoint operators Y which can also be unbounded. To recall how this comes about we start from the general spectral representation of unitary operators,

$$V = \int_{-\pi}^{\pi} \mathrm{e}^{i\varphi}\mathrm{d}E_{\varphi}\,, \tag{6.4}$$

where the projection function E_φ accounts for discrete as well as for continuous contributions. Naive insertion of (6.4) into (6.3) does not generally make sense because $-i(\mathrm{e}^{i\varphi}-1)(\mathrm{e}^{i\varphi}+1)^{-1}=\tan(\varphi/2)$ is not bounded, diverging for $\varphi=\pm\pi$ where the value -1 of the spectrum of V is reached. However, Y is well-defined on Hilbert space vectors f by

$$Yf=\lim_{\tilde{\varphi}\to\pi}\int_{-\tilde{\varphi}}^{\tilde{\varphi}}\tan\frac{\varphi}{2}\mathrm{d}(E_\varphi f) \tag{6.5}$$

in the sense of strong convergence. This is seen noting that with $f=(\mathbb{1}+V)g$ one gets $\tan^2(\varphi/2)\mathrm{d}||E_\varphi f||^2 = 4\sin^2(\varphi/2)\mathrm{d}||E_\varphi g||$ over which the integral from $-\pi$ to π is obviously finite.

Thus with unbounded operators Y in (6.5) we indeed get unitary operators V in (6.4) with a spectrum extending to -1, as is necessary for proper working of the sum rule (3.8) and to guarantee correct results in (3.9). It is seen that for this a Hilbert space is necessary, which not only has infinite dimension but also includes its limit elements. Obviously the latter here is of crucial importance.

Since the projection function E_φ is common to (6.4) and (6.5), we can trace back for V as well as for Y to the projections $P_1^{(\pm)}$ and $P_2^{(\pm)}$ (related to the eigenvalues $+1$ and -1 of V, respectively) which only enter the sum rule. With E_φ being strongly continuous from the right, $\lim_{\varepsilon\to 0}E_{\varphi+0}=E_\varphi$, for this one has the relations $\lim_{\varepsilon\to 0}(E_0-E_{-\varepsilon})=P_1^{(+)}+P_1^{(-)}$ and $\lim_{\varepsilon\to 0}(E_\pi-E_{\pi-\varepsilon})=P_2^{(+)}+P_2^{(-)}$.

To see how the Hilbert space of interest arises in the continuum limit, one has to reconsider things from the appropriate point of view: On the infinite lattice one gets the Hilbert space of sequences $\mathbf{l}_2$ (in a basis of which a vector is related to a lattice site, a Dirac index and a gauge field index). The unitarily equivalent Hilbert space $\mathbf{L}_2(\pi;k)$ of functions $f(k)$ with $-\pi\le k_\mu\le\pi$ (and Dirac and gauge-group indices being suppressed) is obtained from $\mathbf{l}_2$ by a Fourier transformation. Introducing the lattice spacing a and variables $p=k/a$ the space $\mathbf{L}_2(\pi;k)$ becomes $\mathbf{L}_2(\pi/a;p)$. By the limit $a\to 0$ one then gets the operators in $\mathbf{L}_2(\infty;p)$ from the ones in $\mathbf{L}_2(\pi/a;p)$. The space $\mathbf{L}_2(\infty;x)$, unitarily equivalent to $\mathbf{L}_2(\infty;p)$, is again obtained by a Fourier transformation. Instead of proceeding in the more instructive way sketched, one can also realize the direct way from $\mathbf{l}_2$ to $\mathbf{L}_2(\infty;x)$. The equivalent spaces $\mathbf{L}_2(\infty;x)$ and $\mathbf{L}_2(\infty;p)$ are the ones one has in the continuum limit.

In detail the definition of the operators of interest by the indicated limit needs some care. Firstly, since the limit element cannot be given explicitly, we resort to the definition by all matrix elements, i.e. by weak operator convergence. Secondly, because two spaces are involved, this weak limit is to be slightly generalized. To show that this can be properly done, we introduce $f_a(p)=f(p)\Pi_\mu\Theta(\pi/a-p_\mu)$ with $f(p)\in\mathbf{L}_2(\infty;p)$ and the operator $\hat{\mathcal{O}}_a$, requiring $\hat{\mathcal{O}}_a(p',p)$ of $\mathbf{L}_2(\infty;p)$ to be equal to $\mathcal{O}_a(p',p)$ of $\mathbf{L}_2(\pi/a;p)$ for $-\pi/a\le p_\mu\le\pi/a$. Then $\langle f_a|\hat{\mathcal{O}}_a g_a\rangle$ in $\mathbf{L}_2(\infty;p)$ equals $\langle f_a|\mathcal{O}_a g_a\rangle$ in $\mathbf{L}_2(\pi/a;p)$ for all finite a and $\langle f_a|\hat{\mathcal{O}}_a g_a\rangle\to\langle f|\hat{\mathcal{O}}g\rangle$ for $a\to 0$ defines the operator $\hat{\mathcal{O}}$ in $\mathbf{L}_2(\infty;p)$.

Thus, though we cannot give explicit expressions for the operators in $\mathbf{L}_2(\infty;p)$ and $\mathbf{L}_2(\infty;x)$, we can unambiguously deal with them: First calculating the desired matrix elements of the operators or of the functions of operators of interest on the infinite lattice and then performing the $a\to 0$ limit leads to the results in terms of the respective matrix

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elements. It is seen here that what is practice in correct approaches can be precisely formulated and justified in Hilbert space.

The mapping from $\mathcal{O}_a$ to $\hat{\mathcal{O}}$ is not invertible and the spectra of $\mathcal{O}_a$ and $\hat{\mathcal{O}}$ can be substantially different. Obviously the momenta get unbounded in the limit, and this also holds for the operators Y of physical interest. Therefore by the connection between (6.5) and (6.4) the spectrum of V can extend to -1, as has been shown to be necessary for correct results.

We now see how the paradox of ref. [1] resolves: The chiral anomaly there, being already a continuum quantity, should not be considered with the sum rule on the finite lattice but with the one valid in the continuum limit. In this limit the chiral anomaly as well as the sum rule can get correct properties, while on the finite lattice both of them are affected by not reaching the eigenvalue -1 of V. The problems on the finite lattice have been seen to occur not only in the example of ref. [1] but for any well-defined Y.

7. Conclusions and summary

Inspecting possible forms of unitary operators V which define GW Dirac operators, we have pointed out that the Dirac operator proposed by Chiu is based on an operator V of the Cayley-transform type. After a number of necessary preparations, we have analyzed the properties of this class of operators.

Considering the general sum rule for the index we have derived various relations in finite space which have been important in the following. To make sure that the sum rule still holds in the continuum limit, we firstly have shown that the extension to infinite space does not spoil it and secondly that a continuous part of the spectrum does not contribute to it.

We have stressed that the mechanism for admitting a nonvanishing index, exhibited by the sum rule here, is different from that in the Atiyah-Singer case, where instead the space structure itself is allowed to depend on the particular gauge field configuration.

Considering Cayley-type operators V we have seen that for them on the finite lattice the sum rule generally does not allow for a nonvanishing index (and not only in the example of Chiu) because the eigenvalue -1 of V cannot be reached. In addition, we have seen that in case of such V on the finite lattice already the chiral anomaly is affected by this.

Turning to Hilbert space we have pointed out that there the Cayley transformation provides a well-defined connection between unbounded self-adjoint operators and general unitary operators V for which the eigenvalue -1 can be reached. We then have shown that, after introducing a slightly generalized weak operator convergence, the continuum limit precisely provides the appropriate Hilbert-space formulation.

Our results make clear that in the continuum limit the problems of the sum rule for the index and of the chiral anomaly with Cayley-type operators V disappear. The paradox of ref. [1] has been seen to resolve in that the chiral anomaly there, being already a continuum quantity, should not be considered with the sum rule on the finite lattice but with the one valid in the limit.

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